Classification of C^*-algebras admitting ergodic actions of the two-dimensional torus

by

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Abstract

We give a complete classification under *isomorphism of the C^*-algebras which admit an ergodic action of the two-dimensional torus.

1. Ergodic actions of the two-dimensional torus

Let G be the two-dimensional torus, and α a homomorphism of G into $\mathrm{Aut}(A)$ where $\mathrm{Aut}(A)$ is the group of *automorphisms of a C^*-algebra A. We assume $g \to \alpha_g(a)$ to be continuous in the norm on A, $g \in G$, $a \in A$. If α does not act effectively on A, its kernel is a closed subgroup H. It is easy to see that G/H is either the two-dimensional torus or the one-dimensional circle, and in the second case it is easy to see that since G/H acts ergodically $A = \mathbb{C}$, the complex numbers. In the first case G/H is the two-dimensional torus and acts effectively and ergodically. We may therefore just as well assume that α acts effectively on A.

It was proven in [1, Corollary 4.3] that if G acts ergodically on aeffectively on a C^*-algebra A, then $A = A_\rho$ where A_ρ is the C^*-algebra generated by a faithful projective unitary representation

$$\gamma \to a_\gamma \tag{1}$$

of the dual group $Z^2 = \hat{G}$, where

$$a_{\gamma_1} a_{\gamma_2} = c_\rho(\gamma_1, \gamma_2) a \tag{2}$$

and the multiplier c_ρ is given by $c_\rho(\gamma, \tilde{\gamma}) = e^{-2\pi i \lambda n_2 \tilde{n}_1}$ with $\gamma = (n_1, n_2)$ and $\tilde{\gamma} = (\tilde{n}_1, \tilde{n}_2)$ and $\rho = e^{2\pi i \lambda}$. The unitary multiplier representation (2) is implemented by letting a and b be the unitary operators on $L_2(R)$ given by $(af)(x) = f(x+\lambda)$ and $(bf)(x) = e^{2\pi i x} f(x)$ and

$$a_{(n_1, n_2)} = a^{n_1} b^{n_2} \tag{3}$$

Then $ab = \rho ba$ and A_ρ is the C^*-algebra generated by the operator a and b. By interchanging a and b we see that $A_\rho \cong A_{\bar{\rho}}$.

We have the following theorem.

Theorem 1.1

Let α be a continuous ergodic effective action of the two--dimensional torus on a C^*-algebra A. Then $A \cong A_\rho$ where $A_\rho \cong B(L_2(R))$ is the C^*-algebra generated by a and b where $(af)(x) = f(x+\lambda)$, $(bf)(x) = e^{2\pi ix}f(x)$, $\rho = e^{2\pi i\lambda}$ and $\alpha_g(a^{n_1}b^{n_2}) = \gamma(g)a^{n_1}b^{n_2}$, $\gamma = (n_1,n_2)$.

Moreover, if $A_\rho \cong A_\sigma$ then $\rho = \sigma$ or $\rho = \bar{\sigma}$.

This theorem gives a complete classification of the ergodic actions of G on C^*-algebras. The first part of the theorem is as we have already pointed out a consequence of the results in [1]. What remains to be proven is that if $A_\rho \cong A_\sigma$ then $\rho = \sigma$ or $\rho = \bar{\sigma}$. This problem is very different according as ρ is a root of unity or not. If ρ is not a root of unity then the A_ρ are called the irrational rotation algebras. These algebras were studied by M.A. Rieffel [2], M. Pimsner, and D. Voiculescu [3]. From their work it follows that for the irrational rotation algebras $A_\rho \cong A_\sigma$ only if $\rho = \sigma$ or $\rho = \bar{\sigma}$. The method in [3] is by imbedding A_ρ into an AF-algebra and computing K_0 of the AF-algebra. This method utilizes the discrete structure of A_ρ when ρ is not a root of unity, and does not extend to the case when ρ is a root of unity. Hence we need only to consider the case when ρ is a root of unity.

Let therefore ρ be a primitive q-th root of unity, and χ_1 and χ_2 the two generators of the character group $\hat{G} = Z^2$; then the action α_g of G on A_ρ is given by

$$\alpha_g(a^i b^j) = \chi_1(g)^i\ \chi_2(g)^j\ a^i b^j\ , \qquad i,j \in Z \tag{4}$$

where a and b are the generators of A_ρ such that $ab = \rho ba$. We shall now construct A_ρ explicitely as a subalgebra of $C(G) \otimes M_q$ where M_q is the algebra of $q \times q$ matrices, and $C(G)$ the algebra of continuous functions on G. The embedding of A_ρ in $C(G) \otimes M_q$ comes about as follows. G is acting ergodically on the centre C_ρ of A_ρ. For some compact space X, $C_\rho = C(X)$ and there is a group action β of G on X such that for $c \in C_\rho$, $z \in X$

$$\alpha_g(c)(z) = c(\beta_g^{-1}(z))$$

Because of ergodicity G is acting transitively on X. We choose $z_0 \in X$ such that $a^q(z_0) = b^q(z_0) = 1$. We obtain a continuous mapping $h : G \to X$ with $h(g) = \beta_g(z_0)$, and the induced homomorphism

$$h^0 : C_\rho = C(X) \to C(G).$$

Given a character $\chi \in \hat{G}$, χ^q is a function on X, that is, $\chi^q = h^0(k)$ where $k(\beta_g(z_0)) = \chi(g)^q$ or $k(z) = \chi(g)^q$ whenever $\beta_g(z_0) = z$. Using h^0, we form the tensor product $C(G) \underset{C_\rho}{\otimes} A_\rho$.

Setting $K = \chi_1 \otimes a$ and $L = \chi_2 \otimes b$, we have $KL = \rho LK$. We will show that $K^q = L^q = 1$.

Letting $k_1(\beta_g(z_0)) = \chi_1(g)^q$, we have,

$$K^q = \chi_1^q \otimes a^q = h^0(k_1) \otimes a^q = 1 \otimes k_1 a^q.$$

For $z = \beta_g(z_0) \in X$, we have, as $k_1 a^q \in C_\rho$,

$$(k_1 a^q)(z) = \chi_1(g)^q a^q(z) = \chi_1(g)^q \; \alpha_g^{-1}(a)^q(z_0) = \chi_1(g)^q \chi_1(g)^{-q} a^q(z_0) = 1.$$

Thus $K^q = 1$, and similarly, $L^q = 1$.

The subalgebra generated by K and L is isomorphic to $\mathbb{M}_q$; hence, we obtain a homomorphism

$$C(G) \underset{\mathbb{C}}{\otimes} \mathbb{M}_q \xrightarrow{\cong} C(G) \underset{C_\rho}{\otimes} A_\rho$$

which is clearly an isomorphism of $C(G)$-algebras. Writing $A_\rho = 1 \otimes A_\rho$, we have an embedding of A_ρ in $C(G) \underset{\mathbb{C}}{\otimes} \mathbb{M}_q$.

2. Construction of A_ρ

Let $E_1, \ldots, E_q$ be the standard basis of C^q, and let K and L be the matrices with $K(E_i) = \rho^i E_i$ and $L(E_i) = E_{i+1}$ (i taken mod q). We then have $KL = \rho LK$. Let $G = \{(x,y) \in \mathbb{C}^2 \mid |x| = |y| = 1\}$. In the algebra $C(G) \underset{\mathbb{C}}{\otimes} \mathbb{M}_q$, we define $a = x \otimes K$ and $b = y \otimes L$. We then have $ab = \rho ba$ and $a^q = x^q \otimes I_q$ and $b^q = y^q \otimes I_q$. Let T^2 be a torus; $T^2 = \{(u,v) \in C^2 \mid |u| = |v| = 1\}$. Define a homomorphism $d: G \to T^2$ by $d(x,y) = (x^q, y^q)$. We may then set $u = x^q = a^q$ and $v = y^q = b^q$.

The algebra $C(G) \underset{C}{\otimes} \mathbb{M}_q$ is the algebra of sections in the trivial algebra bundle $G \times \mathbb{M}_q$. Let a group H of automorphisms of this bundle be generated by h_1 and h_2 where, for $(x,y,X) \in G \times \mathbb{M}_q$,

$$h_1(x,y,X) = (x, \rho y, KXK^{-1})$$

$$h_2(x,y,X) = (\rho^{-1} x, y, LXL^{-1}) .$$

The group H is of order q^2 and is acting freely on the base G with $G/H = T^2$. Hence

$$pr_\rho : B_\rho = (G \times \mathbb{M}_q)/H \to G/H = T^2$$

is an algebra bundle over T^2 with fibre $\mathbb{M}_q$. The algebra of sections of B_ρ is the H-invariant subspace of $C(G) \underset{\mathbb{C}}{\otimes} \mathbb{M}_q$. The

elements a and b are H-invariant. Any element of $C(G) \otimes \mathbb{M}_q$ may be written uniquely as $\sum_{i,j=0}^{q-1} f_{ij}(x,y)a^i b^j$ and it is invariant if and only if it is of the form $\sum_{i,j=0}^{q-1} h_{ij}(u,v)a^i b^j$.
Noting that $u = a^q$ and $v = b^q$, it follows that the algebra of sections in B_ρ is A_ρ.

3. Automorphisms of A_ρ

The centre of A_ρ is $C(T^2)$, when A_ρ is constructed as above. Any automorphism $\phi\colon A_\rho \to A_\rho$ induces an automorphism $\psi\colon C(T^2) \to C(T^2)$, by restriction. If σ is any other root of unity, an isomorphism $\phi\colon A_\rho \to A_\sigma$ induces an automorphism $\psi\colon C(T^2) \to C(T^2)$, by restriction, as the centre in both algebras has been identified with $C(T^2)$.

Definition 3.1

Given a *automorphism $\psi\colon C(T^2) \to C(T^2)$, let $f\colon T^2 \to T^2$ be the continuous map inducing ψ. Let $\psi' = f_*\colon H_1(T^2) \to H_1(T^2)$. As $H_1(T^2) = \mathbb{Z}^2$; $\psi' \in GL(2,\mathbb{Z})$. Given ψ_i and f_i, $i = 1,2$, the automorphism $\psi_1\psi_2$ is induced by $f_2 f_1$. It follows that $(\psi_1\psi_2)' = \psi_2'\psi_1'$ in $GL(2,\mathbb{Z})$.

Lemma 3.1 Let $\psi\colon C(T^2) \to C(T^2)$ be a *automorphism with $\psi' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then there is a *automorphism $\phi\colon A_\rho \to A_\rho$ with $\phi|C(T^2) = \psi$.

Proof Let $f\colon T^2 \to T^2$ be the homeomorphism inducing ψ. The condition $f_* = \psi' = I_2$ implies that f is homotopic to the identity mapping of T^2. By the homotopy invariance of fibre bundles, there is a bundle automorphism $F\colon B_\rho \to B_\rho$ convering f, that is, there is a commutative diagram,

$$\begin{tikzcd} B_\rho \arrow[r, "F"] \arrow[d, "pr_\rho"'] & B_\rho \arrow[d, "pr_\rho"] \\ T^2 \arrow[r, "f"'] & T^2 \end{tikzcd}$$

where F is an algebra $*$isomorphism in the fibres.
If $s \in A_\rho$, then s is a continuous mapping $s: T^2 \to B_\rho$ with $pr_\rho \circ s = id$. We define $\phi(s) = F^{-1} \circ s \circ f$. Then if $s \in C(T^2)$, $s(u,v)$ is a scalar matrix for each $(u,v) \in T^2$, and we have $\phi(s)(u,v) = F^{-1}s(f(u,v)) = s(f(u,v) = \psi(s)(u,v)$. This shows that ϕ extends ψ. We note that F may be chosen so that ϕ is a $*$automorphism because the structural group of B_ρ by construction is (a subgroup of) $U(q)/T^1$.

<u>Lemma 3.2</u> <u>Given a matrix</u> $M \in GL(2,\mathbb{Z})$. <u>There is a</u> <u>$*$automorphism</u> $\phi: A_\rho \to A_\sigma$ <u>inducing</u> ψ <u>on the centre</u> $C(T^2)$ <u>with</u> $\psi' = M$, <u>and</u> <u>where</u> $\sigma = \rho^{\det M}$.

<u>Proof</u> Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and define ϕ by $\phi(a) = a^\alpha b^\beta$ and $\phi(b) = a^\beta b^\delta$. On the centre ϕ induces $\psi(u) = \phi(a^q) = c_1 a^{\alpha q} b^{\gamma q} = c_1 u^\alpha v^\gamma$ and $\psi(v) = c_2 u^\beta v^\delta$ where $|c_1| = |c_2| = 1$. Thus ψ is induced by the homeomorphism f with $f(u,v) = (c_1 u^\alpha v^\gamma, c_2 u^\beta v^\delta)$. It follows that ϕ is continuous and is defined on all of A_ρ. Also $\psi' = f_* = M$. We must check the relation $ab = \rho ba$. An easy computation shows that $\phi(ab) = \rho^{\alpha\delta - \beta\gamma}\phi(ba)$. We thus have an automorphism $\phi: A_\rho \to A_\rho$ if $\det M = 1$, and an isomorphism $\phi: A_\rho \to A_{\rho^{-1}}$ if $\det M = -1$. In both cases ϕ extends in the obvious way to a $*$automorphism of $C(G) \otimes_{\mathbb{C}} M_q$, showing that ϕ is a $*$isomorphism.

Corollary 3.1 Let $\eta: A_\rho \to A_\sigma$ be a *isomorphism. Then there is a *isomorphsm $\xi: A_\rho \to A_\sigma$ or $\xi: A_\rho \to A_{\sigma^{-1}}$ which is the identity on the centre $C(T^2)$.

Proof Let $(\eta|C(T^2))' = M \in GL(2,\mathbb{Z})$ and let $\phi: A_\sigma \to A_\tau$ be a *isomorphism with $(\phi|C(T^2))' = M^{-1}$, where $\tau = \sigma$ or σ^{-1}. Then $(\phi\eta|C(T^2))' = I_2$, and hence there is a *automorphism $\zeta: A_\rho \to A_\rho$ with $\zeta|C(T^2) = (\phi\eta|C(T^2))^{-1}$. We then put $\xi = \zeta\phi\eta$.

Theorem 3.1 If A_ρ and A_σ are *isomorphic, then $\rho = \sigma$ or $\rho = \sigma^{-1}$.

Proof By Corollary 3.1, there is a *isomorphism $A_\rho \to A_\tau$ which is the identity on the centre and where $\tau = \sigma$ or $\tau = \sigma^{-1}$. Any such isomorphism is induced by an isomorphism of algebra bundles, $B_\rho \to B_\tau$. The Theorem follows from Proposition 3.1.

Proposition 3.1 If the algebra bundles B_ρ and B_σ are isomorphic, then $\rho = \sigma$.

The proof of this Proposition will occupy the remaining pages. Given an algebra bundle B over T^2 with fibre $\mathbb{M}_q$, we will define a complex number $\omega(B)$ with $\omega(B)^q = 1$, and which depends only on the topological properties of B. The number $\omega(B)$ will determine B completely, but we do not need this fact. The problem is to actually compute $\omega(B_\rho)$ as a function of ρ.

Let $S^1 = \{v \in \mathbb{C} \mid |v| = 1\}$ and define the covering projection $E: \mathbb{R} \times S^1 \to T^2$ by $E(s,v) = (e^{2\pi i s}, v)$. A trivialization of the induced algebra bundle $E^*(B)$ over $\mathbb{R} \times S^1$ is an algebra bundle isomorphism $F: \mathbb{R} \times S^1 \times \mathbb{M}_q \to E^*(B)$. We will make an explicit construction of a trivialization for $B = B_\rho$.

Definition of $\omega(B)$. Consider the diagram:

$$
\begin{tikzcd}
\mathbb{R}\times S^1\times \mathbb{M}_q \arrow[r, "F"] \arrow[dr] & E^*(B) \arrow[r, "pr_2"] \arrow[d, "pr_1"] & B \arrow[d, "pr_B"] \\
 & \mathbb{R}\times S^1 \arrow[r, "E"] & T^2
\end{tikzcd}
\tag{3.2}
$$

Here pr_1 and pr_2 are the projections from the fibered product $E^*(B)$ and pr_B is the bundle projection of B. Let $\theta(s,v) = (s+1,v)$ be the covering transformation of E, and let $\theta': E^*(B) \to E^*(B)$ be the induced automorphism of $E^*(B)$ with $pr_1\theta' = \theta pr_1$ and $pr_2\theta' = pr_2$. We obtain an automorphism Θ of $\mathbb{R}\times S^1\times \mathbb{M}_q$ defined by $\Theta = F^{-1}\theta' F$. The group of algebra automorphis of $\mathbb{M}_q$ is $GL(q,\mathbb{C})/\mathbb{C}^* = SL(q,\mathbb{C})/Z_q$ where $Z_q = \{zI_q \mid z^q=1\}$ is the centre of $SL(q,C)$. There is a continuous map

$$\alpha : \mathbb{R}\times S^1 \to SL(q)/Z_q$$

such that

$$\Theta(s,v,X) = (s+1,v,\alpha(s,v)X\,\alpha(s,v)^{-1})$$

The homotopy class of α is an element $[\alpha] \in \pi_1(SL(q)/Z_q) = Z_q$. We define $\omega(B)$ by $[\alpha] = \omega(B)^{-1}I_q$ when viewing $[\alpha]$ as an element of Z_q. We will show that $\omega(B)$ is well defined.

Lemma 3.3 Let $F_1,F_2: \mathbb{R}\times S^1\times \mathbb{M}_q \to E^*(B)$ be two trivializations of $E^*(B)$. Let Θ_i and α_i $(i=1,2)$ be defined as above, using F_i. Then $[\alpha_1] = [\alpha_2]$.

Proof There is a bundle automorphism D of $\mathbb{R}\times S^1\times \mathbb{M}_q$ such that $F_2 = F_1 D$. There is a continuous mapping $\beta: \mathbb{R}\times S^1 \to SL(q)/Z_q$ such that

$$D(s,v,X) = (s,v,\beta(s,v)\,X\,\beta(s,v)^{-1}).$$

By definition, $\theta_2 = F_2^{-1}\theta' F_2 = D^{-1}(F_1^{-1}\theta' F_1)D = D^{-1}\theta_1 D$, and hence, $D\theta_2 = \theta_1 D$. It follows that

$$D(s+1,v,\alpha_2(s,v)\,X\,\alpha_2(s,v)^{-1})$$
$$= \theta_1(s,v,\beta(s,v)\,X\,\beta(s,v)^{-1}) ,$$

and hence

$$(s+1,v,\beta(s+1,v)\alpha_2(s,v)\,X\,\alpha_2(s,v)^{-1}\beta(s+1)v)^{-1}) =$$
$$(s+1,v,\alpha_1(s,v)\beta(s,v)\,X\,\beta(s,v)^{-1}\alpha_1(s,v)^{-1}) \qquad \text{for all } X \in \mathbb{M}_q .$$

It follows that

$$\alpha_2(s,v) = \beta(s+1,v)^{-1}\alpha_1(s,v)\beta(s,v) .$$

A homotopy is given by

$$\gamma_c(s,v) = \beta(s+c-1,v)^{-1}\alpha_1(s,v)\beta(s,v) , \qquad 1 \leq c \leq 2 .$$

We have, in the abelian group $\pi_1(SL(q)/Z_q)$,

$$[\alpha_2] = [\gamma_2] = [\gamma_1] = [\beta]^{-1} [\alpha_1] [\beta] = [\alpha_1] .$$

<u>Proposition 3.2</u> <u>Let</u> $\rho = e^{2\pi ip/q}$ <u>where</u> $(p,q) = 1$. <u>Then</u> $\omega(B_\rho) = e^{2\pi ip'/q}$ <u>where</u> $pp' \equiv 1 \pmod q$.

<u>Proof</u> We shall construct a trivialization of $E^*(B_\rho)$. To this end, we define a bundle mapping

$$F_1 : \mathbb{R}^2 \times \mathbb{M}_q \to G \times \mathbb{M}_q ,$$

$$F_1(s,t,X) = (e^{-2\pi is/q}, e^{2\pi it/q}, e^{2\pi it\kappa}\,X\,e^{-2\pi it\kappa}) ,$$

where κ is the q by q matrix with $\kappa(E_j) = (j/q)E_j$. We have $e^{2\pi iq\kappa} = I_q$ and $e^{2\pi ip\kappa} = K$.

We compute, to obtain

$$h_1 F_1(s,t,X) = (e^{-2\pi is/q}, e^{2\pi i(t+p)/q}, e^{2\pi i(t+p)\kappa}\,X\,e^{-2\pi i(t+p)\kappa}) = F_1(s,t+p,X),$$

and $F_1(s,t+q,X) = F_1(s,t,X)$.

Let F_2 be the conposite bundle mapping,

$$F_2 : \mathbb{R}^2 \times \mathbb{M}_q \xrightarrow{F_1} G \times \mathbb{M}_q \to (G \times \mathbb{M}_q)/H = B_\rho .$$

As $(p,q) = 1$, we have

$$F_2(s,t,X) = F_2(s,t+1,X) .$$

It follows that there is a bundle mapping

$$F_3 : \mathbb{R} \times S^1 \times \mathbb{M}_q \to B_\rho$$

such that $F_3(s,v,X) = F_2(s,t,X)$ when $e^{2\pi it} = v$. F_3 covers the mapping $E: \mathbb{R} \times S^1 \to T^2$ in the base spaces, and hence induces a bundle isomorphism $F: \mathbb{R} \times S^1 \times \mathbb{M}_q \to E^*(B_\rho)$. Now we compute the bundle automorphism $\theta = F^{-1}\theta' F$. Given $(s,v,X) \in \mathbb{R} \times S^1 \times \mathbb{M}_q$, we have, for some $Y \in \mathbb{M}_q$,

$$\theta(s,v,X) = (s+1,v,Y) .$$

Because $pr_2\theta' = pr_2$ (see diagram 3.2), we have $pr_2 F\theta = pr_2\theta' F = pr_2 F$. It follows that $F_3\theta = F_3$ because $F_3 = pr_2 F$. Letting $v = e^{2\pi it}$, this implies that

$$F_2(s,t,X) = F_2(s+1,t,Y) .$$

We note that

$$F_1(s+1,t,Y) = (e^{-2\pi i(s+1)/q},\ e^{2\pi it/q},\ e^{2\pi it\kappa} Y e^{-2\pi it\kappa})$$

$$= (e^{-2\pi is/q}\rho^{-p'},\ e^{2\pi it/q},\ e^{2\pi it\kappa} Y\, e^{-2\pi it\kappa})$$

$$= h_2^{p'}(e^{-2\pi is/q},\ e^{2\pi it/q},\ L^{-p'} e^{2\pi it\kappa} Y\, e^{-2\pi it\kappa} L^{p'})$$

$$= h_2^{p'} F_1(s,t,\alpha(t)^{-1}\, Y\, \alpha(t))$$

where $\alpha(t) = e^{-2\pi it\kappa}\, L^{p'}\, e^{2\pi it\kappa}$.

It follows that

$$F_2(s+1,t,Y) = F_2(s,t,\alpha(t)^{-1} Y \alpha(t)) ,$$

$$F_2(s,t,X) = F_2(s+1,t,\alpha(t) X \alpha(t)^{-1}) ,$$

and hence

$$\theta(s,v,X) = (s+1,v,\alpha(t) X \alpha(t)^{-1})$$

when $v = e^{2\pi it}$.

By definition of $\omega(B_\rho)$, we have

$$\omega(B_\rho)^{-1} I_q = \alpha(1)\alpha(0)^{-1} = e^{-2\pi i\kappa} L^{p'} e^{2\pi i\kappa} L^{-p'}$$

$$= K^{-p'} L^{p'} K^{p'} L^{-p'} = \rho^{-(p')^2} I_q = e^{-2\pi ip'/q} I_q ,$$

and hence, $\omega(B_\rho) = e^{2\pi ip'/q}$.

<u>Proof of Proposition 3.1</u> If B_ρ and B_σ are isomorphic algebra bundles, ρ and σ will both be primitive q^{th} roots of unity, and $\omega(B_\rho) = \omega(B_\sigma)$, since $\omega(B)$ is clearly a topological invariant. If $\rho = e^{2\pi ip_1/q}$ and $\sigma = e^{2\pi ip_2/q}$, we obtain $e^{2\pi ip_1'/q} = e^{2\pi ip_2'/q}$, and hence $p_1' \equiv p_2' (\mathrm{mod}\, q)$. It follows that $p_1 \equiv p_2 (\mathrm{mod}\, q)$, and hence that $\rho = \sigma$.

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